ON-SEMI-MARKOV-ANALOGUE OF RANDOM WALK WITH ABSORBING BARRIERS

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SUMMARY

Some problems pertaining to a random walk with absorbing barriers are dealt with; the time between any two transitions i.e. between a shift in position, being a random variable whose distribution depends on the direction of the movement. Probability of position at time t before absorption, probability of ultimate absorption at a barrier conditioned on the starting position are obtained explicitly. The probability distribution of the duration of the walk has also been discussed. The problem is motivated by questions pertaining to growth of a cancer tumor and multigene evolution.

Introduction and Basic Symbols

Consider a random walk $(t, Z(t))$, in continuous time t, and defined by

$$Z(t) = \sum_{k=0}^{N(t)} X_k, \quad t > 0$$

$Z(0) = X_0$, initial position, let it be at i. $N(t)$ denotes the total number of movements undergone during $(0, t)$ and $\{x_k, k > 0\}$ is a sequence of mutually independent, identically distributed random variable and for $k \geq 0$

$$X_k = \pm 1$$

with

$$P_r(X_k = +1) = p, \; P_r(X_k = -1) = q(= 1 - p)$$

For $k = 0$, the distribution is arbitrary. Let θ_k be the time between the

$(k-1)$th movement and the kth movement. θ_k i.e. $k \geqslant 1$ with $\theta_0 \equiv 0$, are random variables and are assumed to be independently distributed and the distribution functions depend on the direction of the movement i.e. when the shift from the present position is to the right or to the left neighbouring position.

Let

$$F_{k-1,k}(t) = F_1(t), \text{ if the movement is to the right}$$
$$= F_2(t) \text{ otherwise.}$$

The random walk defined above may also be called as Semi-Markov Random Walk (S.M.R.W.) This random walk is studied in the presence of absorbing barriers placed at "b" and "a", the position being determined by considering a fixed frame of reference. Thus, the movement of the walk is restricted to the set of integers

$$\{-b, -b+1, \ldots, -1, 0, 1, \ldots, a-1, a\},$$

which is also called the state space of the process. Some problems like the probability of the position of the Walk at time t before absorption, probability of ultimate absorption at a barrier and the probability distribution of the duration of the walk are studied.

As stated by Beyer and Waterman [1] the walk with absorbing barriers can model the growth of a cancer tumor and multigene evolution. Thus the explicit results obtained for the S.M.R.W. with absorbing barriers are useful.

Further in the notation of Pyke [5] and [6], $Q_{ij}(t)$ i.e. the probability that after making a transition into state i, the process makes next transition into state j, in an amount of time less than or equal to t, is given by

$$Q_{ij}(t) = \begin{cases} pF_1(t) & \text{if } j = i+1 \\ qF_2(t) & \text{if } j = i-1 \\ o & \text{Otherwise} \end{cases}$$

and

$$H_i(t) = \sum_j Q_{ij}(t) = p F_1(t) + q\, F_2(t),$$

which gives the probability of transition from position i to any of the possible positions in one step.

Further suppose that the Laplace-Stieltjes Transform (L.-S.T.) of the various functions are obtained by their corresponding small letters, say $q_{ij}(s)$, $f_k(s)$, $h_i(s)$ respectively are L.-S.T. of $Q_{ij}(t)$, $F_k(t)$ and $H_i(t)$.

2. Probability of Position at Time *t* before Absorption

Let $P_{ij}(t, n)$ be the probability that the walk is at position j at time t in n transitions before absorption occurs given that the walk started at position i. Also suppose $P_{ij}(t)$ is the probability that the position at time t is j before absorption occurs given that the initial position was i.

Thus,

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}(t, n)$$

For $-b < i, j < a$ by relating the position of the walker at the nth transition with its possible positions at the $(n-1)$ th transition, n being the number of transitions in time t, we have

$$P_{ij}(t, n) = P_{i,j-1}(t, n-1) * \theta_{j-1,j}(t) + P_{i,j+1}(t, n-1) * \theta_{j+1,j}(t) \quad \text{for } n > 0 \qquad (2.1)$$

$$P_{ij}(t, 0) = \delta_{ij}(1 - H_i(t)) \qquad (2.2)$$

Where δ_{ij} is a Kronecker delta and $*$ stands for convolution here.

The appropriate boundary conditions are

$$P_{-b,i}(t, n) = 0 = P_{a,i}(t, n) \text{ for } i > 0 \text{ and } -b < i < a \qquad (2.3)$$

Equations (2.1) to (2.3) yield

$$P_{ij}(t) = \delta_{ij}(1 - H_i tt)) + \sum_{k=j-1,j+1} \int_0^t P_{ik}(t-\tau)\, d\theta_{kj}(\tau) \qquad (2.4)$$

with

$$P_{-b,j}(t) = 0 = P_{a,j}(t) \text{ for } t > 0 \text{ and } -b < j < a \qquad (2.5)$$

Substituting $Q_{kj}(t)$ and $H_i(t)$ and then on taking L.-S.T. with respect to t, we have

$$p_{ij}(s) = \delta_{ij}(1 - pf_1(s) - qf_2(s)) + pf_1(s)\, p_{i,j-1}(s) + qf_2(s)\, p_{i,j+1}(s) \qquad (2.6)$$

with

$$p_{-b,j}(s) = 0 = p_{a,j}(s) \qquad (2.7)$$

Alternatively, these can be written in the vector form as

$$\mathbf{p}_j(s) = (1 - pf_1(s) - qf_2(s)\, \mathbf{e}_j + pf_1(s)\, \mathbf{p}_{j-1}(s) + qf_2(s)\, \mathbf{p}_{j+1}(s) \qquad (2.8)$$

with

$$\mathbf{p}_{-b}(s) = 0 = p_a(s) \tag{2.9}$$

The vectors are column vectors. Further $\mathbf{e}_j$ denotes a unit column vector whose jth element is one and all other elements zero. The equations (2.8) can be written as

$$(1 - pf_1(s)\, E^{-1} - qf_2(s)\, E)\, \mathbf{p}_j(s) = (1 - pf_1(s) - qf_2(s))\, \mathbf{e}_j$$

where E is the difference operator. A particular solution of the difference equation is given by

$$\mathbf{p}_j(s) = (1 - pf_1(s) - qf_2(s)) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (pf_1(s))^{n-k} (qf_2(s))^k \cdot \mathbf{e}_{2k-n+j}$$

where as the ith element of $\mathbf{p}_j(s)$ is

$$p_{ij}(s) = (1 - pf_1(s) - qf_2(s) \sum_{n=|i-j|, |i-j|+2, \ldots} \binom{n}{\frac{n+i-j}{2}} \times (pf_1(s))^{\frac{n-i+j}{2}} (qf_2(s))^{\frac{n+i-j}{2}} \tag{2.10}$$

Case (*i*) : $i \geqslant j$

Putting $r = (n + j - i)/2$ in equation (2.10), we get

$$p_{ij}(s) = (1 - pf_1(s) - qf_2(s)^{i-j} \sum_{r=0}^{\infty} \binom{2r + i - j}{r} (A(s))^r \tag{2.11}$$

$$\text{where} \quad A(s) = pqf_1(s)\, f_2(s) \cdot \tag{2.12}$$

Using the identify

$$\sum_{k=0}^{\infty} \binom{\alpha + \beta k}{k} Z^k = \frac{x^{\alpha+1}}{(1-\beta)\, x + \beta}, \text{ Gould [4]} \tag{2.13}$$

where

$$Z = \frac{x-1}{x^{\beta}}, \; |Z| = \left| \frac{(\beta-1)^{\beta-1}}{\beta^{\beta}} \right|$$

we have

$$p_{ij}(s) = (1 - pf_1(s) - qf_2(s))\,(qf_2(s))^{i-j}\cdot\frac{(x(s))^{(i-j+1)}}{x(s)+2} \tag{2.14}$$

where

$$x(s) = \frac{\alpha(s)}{2A(s)},\ \frac{\beta(s)}{2A(s)}$$

and

$$\alpha(s),\beta(s) = 1 \pm (1-4A(s))^{1/2}.$$

Case (ii) : $i \leqslant j$

Putting $r = (n-j+1)/2$ in equation (2.10) and by using the identify (2.13) similarly, we have

$$p_{ij}(s) = (1 - pf_1(s) - qf_2(s))\,(pf_1(s)^{j-i}\,x(s))^{j-i+1}/(x(s)+2) \tag{2.15}$$

The general solution of the equation (2.6) is

$$p_{ij}(s) = C^{(1)}_{(i)}(s)\cdot\frac{(\alpha(s)^j)}{(2qf_2(s))^j} + C^{(2)}_{(i)}(s)\,\frac{(\beta(s))^j}{(2qf_2(s))^j} + (1 - pf_1(s)$$
$$- qf_2(s))\,(qf_2(s))^{i-j}\cdot\frac{(x(s))^{(i-j+1}}{x(s)+2},\ \text{for } i \geqslant j \tag{2.16}$$

$$= C_i^{(1)}(s)\,\frac{(\alpha(s))^j}{(2qf_2(s))^j} + C_i^{(2)}(s)\cdot\frac{(\beta(s))^j}{(2qf_2(s))^j} + (1 - pf_1(s)$$
$$- qf_2(s))\,(qf_2(s))^{j-i}\cdot\frac{(x(s))^{j-i+1}}{x(s)+2},\ \text{for } i \leqslant j \tag{2.17}$$

Using the boundary conditions (2.7), we get

$$C_i^{(1)}(s) = \frac{x(s)\,(1 - pf_1(s) - qf_2(s))}{(x(s)+2)\,(\alpha^{-b}(s)\cdot\beta^a(s) - \alpha^a(s)\,\beta^{-b}(s)} \times [(pf_1(s))^{a-i}\cdot$$
$$(2qf_2(s))^a\cdot\beta^{-b}(s)\cdot(x(s))^{a-i} - (qf_2(s))^i\cdot 2^{-b}\cdot\beta^a(s)\cdot$$
$$(x(s))^{b+i}]. \tag{2.18}$$

$$C_i^{(2)}(s) = \frac{x(s)}{(x(s)+2)}\cdot\frac{(1 - pf_1(s) - qf_2(s))}{(\alpha^a(s)\,\beta^{-b}(s) - \alpha^{-b}(s)\,\beta^a(s))} \times$$
$$[(pf_1(s)^{a-i}\cdot(2qf_2(s))^a\cdot\alpha^{-b}(s)\cdot(x(s)^{a-i} - (qf_2(s))^i$$
$$\cdot 2^{-b}\cdot\alpha^a(s)\cdot(x(s)^{b+i}] \tag{2.19}$$

This result is corresponding to the result (72), in Cox and Miller ((2), pp. 54), concerning classical Random Walk. We verify that, if $f_1(s) = e^{-s} = f_2(s)$, on inversion we get the result of the Classical Random Walk

3. Probability of Absorption

Let

$k_{ia}(i, n)\, dt = p_r$ [the walk is at barrier 'a' for the first time during $(t, t+dt)$; $N(t) = n \mid Z(0) = i$]

and $k_{ia}(t)\, dt = p_r$ [the walk is at barrier 'a' for the first time during

$$(t, t+dt) \mid Z(0) = i]$$

$$= \sum_{n=0}^{\infty} k_{ia}(t, n)\, dt \tag{3.1}$$

Also

$$\left.\begin{aligned} k_{i,a}(t, 0) &= 0, \text{ for } -b < i < a \\ k_{-b,a}(t, n) &= 0, n \geqslant 0 \\ k_{a,a}(t, 0) &= \delta(t) \\ k_{a,a}(t, n) &= 0, n \geqslant 1 \end{aligned}\right\} \tag{3.2}$$

Where $\delta(t)$ is Dirac function. For $-b < i < a$, we have

$$k_{i,a}(t, n) = \int_0^t k_{i+1,a}(t-\tau, n-1)\, pf^{(1)}(\tau) d\tau$$

$$+ \int_0^t k_{i-1,a}(t-\tau, n-1)\, qf^{(2)}(\tau) d\tau \tag{3.3}$$

Where $f^{(1)}(t)$ and $f^{(2)}(t)$ are probability density functions (p.d.fs.) corresponding to the cummulative distribution functions (c. d. fs.) $F_1(t)$ and $F_2(t)$ respectively.

Summing over n and then taking Laplace Transform (L. T.) with respect to t on both the sides, we get for $-b < i < a$

$$k^*_{i,a}(s) = k^*_{i+1,a}(s) \cdot pf^*_1(s) + k^*_{i-1,a}(s) \cdot qf^*_2(s), \tag{3.4}$$

where $k^*_{ia}(s)$, $f^*(s)$ and $f^*_2(s)$ are respectively L. T. of $k_{ia}(t)$, $f^{(1)}(t)$ and $f^{(2)}(t)$. The appropriate boundary conditions are

$$k^*_{a,a}(s) = 1,\ k^*_{-b,a}(s) = 0 \tag{3.5}$$

The general solution of (3.4) can easily be obtained as

$$k^*_{i,a}(s) = \frac{Z_1^{i+b}(s) - Z_2^{i+b}(s)}{Z_1^{a+b}(s) - Z_2^{a+b}(s)} \tag{3.6}$$

where

$$Z_1(s), Z_2(s) = [1 \pm (1 - 4\, B(s))^{1/2}]/2pf_1^*(s)$$

and

$$B(s) = pqf_1^*(s)\, f_2^*(s)$$

Similarily, we have

$$k^*_{i,-b}(s) = \frac{Z_1^{a+b}(s)\, Z_2^{b+i}(s) - Z_1^{b+i}(s)\, Z_2^{a+b}(s)}{Z_1^{a+b}(s) - Z_2^{a+b}(s)} \tag{3.7}$$

The probability of ultimate absorption at 'a' starting from the position i, can be given by

$$k^*_{i,a}(0) = \frac{1 - (q \mid p)^{b+i}}{1 - (q \mid p)^{a+b}},\ -b \leqslant i \leqslant a,\ p \neq q \tag{3.8}$$

$$= (b + i)/(a + b),\ -b \leqslant i \leqslant a,\ p = q$$

And, the probability of ultimate absorption at the barries '$-b$' is given by

$$k^*_{i,-b}(0) = \frac{(q \mid p)^{b+i} - (q \mid p)^{a+b}}{1 - (q \mid p)^{a+b}},\ -b \leqslant i \leqslant a \text{ and } p \neq q \tag{3.9}$$

$$= 1 - \frac{b+i}{a+b} = \frac{a-i}{a+b},\ -b \leqslant i \leqslant a \text{ and } p = q$$

We note that

$$k^*_{ia}(0) + k^*_{i,-b}(0) = 1 \tag{3.10}$$

which demonstrates that the S. M. R. W. in the presence of two absorbing barriers terminates with probability one.

If $b = 0$, the probability of absorption at zero starting from position i, is given by

$$\frac{(q \mid p)^a - (q \mid p)^i}{(q \mid p)^a - 1};\ (\text{for } p \neq q) \quad \text{and} \quad 1 - \frac{i}{a}\ (\text{for } p = q), \tag{3.11}$$

which is also the probability of Gambler's ultimate ruin starting with capital 'i' against adversory with capital 'a' for the Gambler's Ruin Problem, Feller ([3] pp. 345). Thus, it is observed that the probability of ultimate absorption at a barrier does not change even if transitions take random time with any distribution.

4. Expected Duration of the Walk

Let $f(t)$ denote the p.d.f. of the absorption time given that the Walk started from position i, we have

$$f(t) = k_{i,a}(t) + k_{i,-b}(t) \tag{4.1}$$

Taking L. T. and then on substituting $k^*_{i\,a}(s)$ and $k^*_{,-b}(s)$ from equations (3.6) and (3.7), we get

$$f^*(s) = \frac{Z_1^{b+i}(s)\,(1 - Z_2^{a+b}(s)) - Z_2^{b+i}(s)\,(1 - Z_1^{a+b}(s))}{Z_1^{a+b}(s) - Z_2^{a+b}(s)} \tag{4.2}$$

where $f^*(s)$ is L. T. of $f(t)$ and so is the L. T. of the duration of the walk. The expected duration of the walk is given by $-f^{*\prime}(0)$ and this can be easily worked out to yield

$$-f^{*\prime}(0) = \frac{(b+i)\,(p\mu_1 + q\mu_2)}{q-p} - \frac{(a+b)\,(p\mu_1 + q\mu_2)\,\{(1-(q\,|\,p)^{b+i}\}}{(q-p)\,\{1-(q\,|\,p)^{a+b}\}}, \text{ for } p \neq q \tag{4.3}$$

$$= \tfrac{1}{2}(b+i)\,(a-i)\,(\mu_1 + \mu_2), \text{ for } p = q$$

where $\mu_1 = -f^{*\prime}(0)$ and $\mu_2 = -f_2^{*\prime}(0)$ i.e. μ_1 and μ_2 are the mean of the times for moving to right and to left respectively. When $b = 0$ and $\mu_1 = \mu_2 = 1$, then the expected duration of the walk is given by

$$\frac{i}{q-p} - \frac{a}{q-p} \cdot \frac{1-(q/p)^i}{1-(q/p)^a}, \text{ when } p \neq q \tag{4.4}$$

and

$$i(a-i), \text{ when } p = q, \tag{4.5}$$

which is also the expected duration of the classical ruin problem, (Feller [3], pp. 348-49).

Thus, we observe that if the transition time is a random variable with any general distribution, but having unit mean for both the transition directions, the expected duration of the walk is equal to that of the classical ruin problem and is given by (4.4) and (4.5)

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